CRITICAL POINTS OF THE HEAT KERNEL ON A COMPACT SEMISIMPLE LIE GROUP

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A

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By

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Abstract.

Critical points of the heat kernel on a compact semisimple Lie group are studied. The necessary background topics from abstract harmonic analysis, Lie group and Lie algebra theory, and Riemannian geometry are discussed. The fact that the heat kernel is a smooth class function on a compact semisimple Lie group is used to obtain partial results concerning location and degeneracy of its critical points. Original results on critical points of smooth class functions on a compact semisimple Lie group are presented in the last chapter.

Contents

Preface.

The study of a heat kernel on Riemannian manifolds has a long and rich history. The main impetus of such studies comes from the fact that as an analytical object, the heat kernel encodes a lot of geometric and topological information about the manifold, which makes it possible to apply certain analytic techniques in Riemannian geometry.

This work is an attempt to investigate one of the basic properties of a heat kernel as a smooth function on a manifold: the nature of its critical points in case where the manifold is a compact semisimple Lie group.

Unfortunately, even with such a highly symmetric underlying manifold as a semisimple Lie group, only partial results were obtained. Chapter 3 contains original results concerning general behaviour of a critical set of a smooth class function, which shed some light on the Morse properties of the heat kernel.

Chapter 1 contains a very brief exposition of the author's understanding of harmonic analysis on a compact semisimple Lie group and is mainly a review of several chapters in [4] and [5].

Chapter 2 contains a minimum of theory of Lie groups and analysis on Riemannian manifolds necessary to construct the heat kernel on a compact semisimple Lie group.

Chapter 4 is an appendix discussing Poisson summation on flat skew tori. This is a basis for a generalization of Poisson summation for class functions on compact semisimple Lie groups.

1 Some harmonic analysis on Lie groups.

A compact Lie group as a manifold has a large family of natural diffeomorphisms, namely all left and right translations by the group elements. It seems reasonable to study the properties of usual objects studied in analysis (functions, vector fields, differential operators etc.) invariant under these transformations. For an instance, the set of all left-invariant vector fields (i.e. vector fields invariant under all left translations) turns out to be isomorphic to the tangent space of the group at the identity element. Moreover, it is easy to see that the Lie bracket of two left-invariant vector fields is also left-invariant, which endows the tangent space at the identity element with the Lie algebra structure. In other words, left-invariant vector fields form a finite- dimesional Lie subalgebra of the Lie algebra of smooth vector fields on the group and this subalgebra is isomorphic to the tangent space at identity.

Given that one can identify first order differential operators with smooth vector fields on the manifold, the study of invariant higher order operators seems very promising. The first technical problem that arises here is that one needs a coordinate free definition of a differential operator on a manifold.

1.1 Differential operators on manifolds.

Definition. [5] Let $V \subset \mathbf{R}^n$ be an open set and let $\mathcal{D}(V)$ denote the set of all compactly supported smooth functions on V. A *differential operator* on V is a linear mapping $D : \mathcal{D}(V) \to \mathcal{D}(V)$ with the following property:

for each relatively compact open set $U \subset V$ s.t. $\bar{U} \subset V$, there exists a finite family of smooth on U functions a_α $(\alpha = (\alpha_1, \ldots, \alpha_n), \alpha_i \in \mathbf{Z}^+)$ such that

$$D\phi = \sum_\alpha a_\alpha D^\alpha \phi$$

Here $D^\alpha \phi = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} \phi$.

Obviously, differential operator D on V has the property

$$\operatorname{supp}(D\psi) \subset \operatorname{supp}(\psi), \qquad \psi \in \mathcal{D}(V). \tag{1}$$

It follows than, that D can be extended to a linear operator from $\mathcal{E}(V)$ to $\mathcal{E}(V)$ (where $\mathcal{E}(V)$ is a set of all smooth functions on V) by

$$(Df)(x) = (D\phi)(x),$$

where $x \in V$, $f \in \mathcal{E}(V)$ and $\phi \in \mathcal{D}(V)$ coincides with f in a neighborhood of x (clearly the choice of ϕ does not matter).

A less trivial fact, is that any linear operator from $\mathcal{D}(V)$ to $\mathcal{D}(V)$ satisfying (1) is a differential operator.

Theorem. *Let $D : \mathcal{D}(V) \to \mathcal{D}(V)$ is a linear mapping satisfying condition (1). Then D is a differential operator on V. Conversely, any differential operator on V satisfies (1).*

For proof see Chapter II in [5].

This theorem provides one with a way to construct a coordinate free definition of a differential operator on a manifold.

Definition. Let M be a manifold. A *differential operator* D on M is a linear mapping of $C_c^\infty(M)$ into itself which decreases supports:

$$\operatorname{supp}(Df) \subset \operatorname{supp}(f), \qquad f \in C_c^\infty(M).$$

If (U, ϕ) is a local coordinate map on M, the mapping

$$D^\phi : F \to (D(F \circ \phi)) \circ \phi^{-1}, \qquad F \in C_c^\infty(\phi(U)),$$

satisfies the assumption of the previous theorem, therefore for each open relatively compact set W, with $\bar{W} \subset U$ there exists a finite family of functions $a_\alpha \in C^\infty(W)$ such that

$$Df = \sum_\alpha a_\alpha (D^\alpha(f \circ \phi^{-1})) \circ \phi, \qquad f \in C_c^\infty(W).$$

Hence, a differential operator on a manifold M is a usual differential operator in a local coordinate system.

Following subsets of $\mathbf{R}^n$ one can extend D to a linear operator from $C^\infty(M)$ to $C^\infty(M)$.

Now, let $\mathbf{E}(M)$ denote the set of all differential operators on M. Let $\phi : M \to M$ be a diffeomorphism of M. Then define the *image* of $D \in \mathbf{E}(M)$ under ϕ by

$$D^{\phi}(g) = (D(g \circ \phi)) \circ \phi^{-1} \text{ for all } g \in C^{\infty}(M)$$

Definition. Let $\phi : M \to M$ be a diffeomorphism of M, then a differential operator $D \in \mathbf{E}(M)$ is called *invariant under* ϕ if $D^{\phi} = D$, that is

$$Dg = (D(g \circ \phi)) \circ \phi^{-1},$$

or, equivalently,

$$(Dg) \circ \phi = D(g \circ \phi).$$

1.2 Invariant differential operators on a Lie group.

Let G be a Lie group. There is a family of natural automorphisms of G consisting of left translations:

$$L_g : G \to G, \qquad L_g(x) = gx \text{ for all } x \in G$$

Definition. A *left-invariant* differential operator on G is a differential operator D such that

$$D(f \circ L_g) = (Df) \circ L_g$$

or, equivalently,

$$D(f(gx)) = (Df)(gx)$$

for all $f \in C^{\infty}(M)$ and all $g, x \in G$. The algebra of left-invariant differential operators on g is denoted $\mathcal{D}(G)$.

Example. On $\mathbf{R}^n$ and $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ viewed as Abelian groups (with group operation being addition for $\mathbf{R}^n$ and addition $mod\, 1$ on $\mathbf{T}^n$) invariant differential operators are all differential operators with constant coefficients. (∂_i form a basis for invariant vector fields on both. In fact, Lie algebras of $\mathbf{R}^n$ and $\mathbf{T}^n$ are both isomorphic to $\mathbf{R}^n$ with Lie bracket $[X, Y] \equiv 0$.)

It is clear that left-invariant vector fields satisfy this definition, and since one can identify all first order differential operators with smooth vector fields, left-invariant vector fields are all left-invariant first order differential operators. Moreover, it turns out that one can obtain all left-invariant operators on G combining these first order ones. There are two closely connected algebras on $\mathbf{g}$ related to the algebra of all left-invariant operators. They are the universal enveloping algebra of the Lie algebra $\mathbf{g}$ and the symmetric algebra on $\mathbf{g}$.

Let G be compact semisimple Lie group and $\mathbf{g}$ be its Lie algebra. Let $T(\mathbf{g})$ denote the tensor algebra over $\mathbf{g}$ considered as a vector field. Let J be the two sided ideal in $T(\mathbf{g})$ generated by the set of all elements of the form $X \otimes Y - Y \otimes X - [X,Y]$ where $X, Y \in \mathbf{g}$.

Definition. The *universal enveloping algebra* of $\mathbf{g}$ is the factor algebra $U(\mathbf{g}) = T(\mathbf{g})/J$.

The universal enveloping algebra is an associative algebra that contains an isomorphic image of the Lie algebra of G.

Let $X^* \in U(\mathbf{g})$ denote the canonical projection of $X \in \mathbf{g}$.

Theorem. *Let V be a vector space. There is a natural one-to-one correspondence between representations of $\mathbf{g}$ on V and representations of $U(\mathbf{g})$ on V. If ρ is a representation of $\mathbf{g}$ on V and ρ^* is th corresponding representation of $U(\mathbf{g})$, then $\rho(X) = \rho^*(X^*)$ for all $X \in \mathbf{g}$.*

Let $X_1, X_2, \ldots, X_n$ be a basis of $\mathbf{g}$ and define $X^*(t) = \sum_{i=1}^{n} t_i X_i^*$, where $t_i \in \mathbf{R}$. Then for every ordered set of nonnegative integers $m = (m_1, m_2, \ldots, m_n)$ define $X^*(m) \in U(\mathbf{g})$ to be the coefficient of t^m in the expansion of $(|m|!)^{-1}(X^*(t))^{|m|}$, where $|m| = m_1 + m_2 + \ldots + m_n$ and $t^m = t_1^{m_1} t_2^{m_2} \ldots t_n^{m_n}$. Put $X^*(m) = 1$ if $|m| = 0$.

Example. Let X_1, X_2 be a basis for some Lie algebra. Then $X^*(1,1) =$

$(X_1^*X_2^* + X_2^*X_1^*)/2$; $X^*(1,2) = (X_1^*X_2^*X_2^* + X_2^*X_1^*X_2^* + X_2^*X_2^*X_1^*)/3!$.

Proposition. *$U(\mathbf{g})$ as a vector space is spanned by the elements $X^*(m)$ where m varies over all n-tuples of nonnegative integers.*

The main reason why the enveloping algebra is introduced here is because it is isomorphic (as an algebra) to the algebra $\mathcal{D}(G)$ of left-invariant differential operators on G. (See [4].)

Now let $S(\mathbf{g})$ be the symmetric algebra over $\mathbf{g}$.

Theorem. *There exists a unique linear bijection of vector spaces*

$$\lambda : S(\mathbf{g}) \to \mathcal{D}(G)$$

such that $\lambda(X^m) = X^{*m}$ if $X \in \mathbf{g}$ and $m \in \mathbf{Z}^+$. If $X_1, X_2, \ldots, X_n$ is any basis of $\mathbf{g}$ and $P \in S(\mathbf{g})$, then

$$(\lambda(P)f)(g) = [P(\partial_1, \ldots, \partial_n) f(g \exp(t_1 X_1 + \ldots + t_n X_n))]_{t=0},$$

where $f \in C^\infty(G)$, $\quad \partial_i = \partial/\partial t_i$, $t = (t_1, \ldots, t_n)$, and $g \in G$.

Definition. The mapping λ is usually called *symmetrization.*

Existence of symmetrization proves that symmetric algebra and universal enveloping algebra are closely related. In fact, in case of abelian Lie group it's easy to see that they are isomorphic as algebras.

It is worth noticing that if $Y_1, \ldots, Y_p \in \mathbf{g}$, then

$$\lambda(Y_1 \cdots Y_p) = \frac{1}{p!} \sum_{\sigma \in S_p} Y^*_{\sigma(1)} \cdots Y^*_{\sigma(p)}$$

where S_p is the symmetric group on p letters.

1.3 Harmonic analysis on a Lie group.

A function on G which is an eigenfunction of every $D \in \mathcal{D}(G)$ is called a *joint eigenfunction* of $\mathcal{D}(G)$. Given a homomorphism

$$\chi : \mathcal{D}(G) \to \mathbf{C}$$

the space

$$E_\chi(G) = \{f \in C^\infty(G) : Df = \chi(D)f \quad \text{for all } D \in \mathcal{D}(G)\}$$

is called the *joint eigenspace* for χ.

Let T_χ denote the natural representation of G on $E_\chi(G)$, that is $(T_\chi(g)f)(h) = f(g^{-1}h)$ for all $f \in E_\chi(G)$ and $h \in G$. These representations are called *eigenspace representations.*

According to [5], harmonic analysis on a Lie group G is broadly speaking a study of the following questions:

- Is there a way to decompose functions on G into joint eigenfunctions of $\mathcal{D}(G)$ and what functions admit such a decomposition?
- What are the joint eigenspaces of $\mathcal{D}(G)$?
- For which χ is the eigenspace representation T_χ of G irreducible?

Example. On $\mathbf{R}^n$ the algebra of left-invariant operators consists of all differential operators with constant coefficients. Then, joint eigenfunctions in this case are the exponential functions $f(x_1, \ldots, x_n) = \exp(c_1x_1 + \ldots + c_nx_n)$, where $c_i \in \mathbf{C}$. Let $D = P(\partial_1, \ldots, \partial_n)$ (where P is a polynomial in n variables) be an element of $\mathcal{D}(\mathbf{R}^n)$. Then $\chi(D) = P(c_1, \ldots, c_n)$ for some set of complex constants $c_1, \ldots, c_n$ corresponds to a one-dimensional joint eigenspace $E_\chi(\mathbf{R}^n) = \{C\exp(c_1x_1 + \ldots + c_nx_n) : C \in \mathbf{C}\}$. It follows that in this case complete answers to the questions posed can be given:

- The joint eigenfunction decomposition becomes in this case a usual Fourier transform on $\mathbf{R}^n$.
- Joint eigenspaces are one-dimensional, each spanned by some $\exp(c_1x_1 + \ldots + c_nx_n)$.
- Since all joint eigenspaces are one-dimensional, eigenspace representations are irreducible.

Example. On $\mathbf{T}^n$ the situation is very similar, the only difference being that $c_i \in \mathbf{Z}$. Then Fourier transform becomes a Fourier series decomposition, and instead of uncountable collection of one-dimensional joint eigenspaces, there are only countable many.

This difference in cardinality of the collection of joint eigenspaces is a reflection of crucial distinction between $\mathbf{R}^n$ and $\mathbf{T}^n$, namely, the second is compact and the first is not.

In this context, the famous Peter-Weyl theorem gives an answer to the first question in case of a compact Lie group. Here instead of $\mathcal{D}(G)$ one considers a narrower algebra of operators invariant under both left and right translations, called *bi-invariant operators.*

Proposition. *Assume G is a connected Lie group. Let $\mathcal{Z}(G)$ the center of $\mathcal{D}(G)$. Then $\mathcal{Z}(G)$ consists of right invariant differential operators in $\mathcal{D}(G)$, in other words, of bi-invariant operators on G*

Let $\alpha \in [\alpha] \in \hat{G}$ be a representative of an equivalence class of irreducible representations of G and suppose it acts on some vector space V_α. Let d_α be the dimension of V_α. From now on suppose some basis is chosen and fixed in V_α. Now $\alpha : G \to GL(V_\alpha)$, so let $M^\alpha_{ij}(g)$ denote the elements of the matrix $\alpha(g), \quad g \in G$ with respect to the chosen basis. For every pair i, j such that $1 \leq i, j \leq d_\alpha$, $M^\alpha_{ij}(g)$ is a continuous function on G.

Proposition. *For all i, j such that $1 \leq i, j \leq d_\alpha$ and all irreducible representations α, $M^\alpha_{ij}(x)$ are joint eigenfunctions of $\mathcal{Z}(G)$. For a fixed α, $M^\alpha_{ij}(x)$ belong to the same joint eigenspace for all $1 \leq i, j \leq d_\alpha$.*

Proof. Let $D \in \mathcal{Z}(G)$ be a bi-invariant operator and let $\varphi^\alpha_{i,j}(x) = DM^\alpha_{ij}(x)$. Then if $x, y \in G$

$$\varphi^\alpha_{i,j}(xy) = (DM^\alpha_{ij})(xy) = D(M^\alpha_{ij}(xy)),$$

since D is right invariant. Now

$$M_{ij}^{\alpha}(xy) = \sum_{k=1}^{d_\alpha} M_{ik}^{\alpha}(x) M_{kj}^{\alpha}(y),$$

because α is a representation, and therefore $M(xy) = M(x)M(y)$ where the multiplication on the right hand side is a matrix multiplication. Combination of the last two formulas implies

$$\varphi_{ij}^{\alpha}(xy) = \sum_{k=1}^{d_\alpha} \varphi_{ik}^{\alpha}(x) M_{kj}^{\alpha}(y).$$

Similarly, using left-invariance of D one gets

$$\varphi_{ij}^{\alpha}(yx) = \sum_{k=1}^{d_\alpha} M_{ik}^{\alpha}(y) \varphi_{kj}^{\alpha}(x).$$

Substituting the group identity e for x gives

$$\varphi_{ij}^{\alpha}(y) = \sum_{k=1}^{d_\alpha} \varphi_{ik}^{\alpha}(e) M_{kj}^{\alpha}(y) = \sum_{k=1}^{d_\alpha} M_{ik}^{\alpha}(y) \varphi_{kj}^{\alpha}(e),$$

or in matrix notation $\varphi^\alpha(e)M(y) = M(y)\varphi^\alpha(e)$ and thus by irreducibility of α and Schur's Lemma $\varphi^\alpha(e) = \lambda_\alpha E_{d_\alpha}$, where E_{d_α} as the identity matrix of size d_α. Therefore

$$\varphi_{ij}^{\alpha}(y) = DM_{ij}^{\alpha}(y) = \lambda_\alpha M_{ij}^{\alpha}(y)$$

for all $y \in G$. $\square$

Theorem. (Peter-Weyl) *Let G be compact Lie group. The set of finite linear combinations of $M_{ij}^{\alpha}(g)$ for all $[\alpha] \in \hat{G}$ and all $1 \le i, j \le d_\alpha$ is dense in the $\| \bullet \|_\infty$ norm in $C(G)$, the set of all continuous functions on G.*

Theorem. (Orthogonality relations). *For any α, β such that $[\alpha], [\beta] \in \hat{G}$ and $1 \le i, j \le d_\alpha$; $1 \le k, l \le d_\beta$:*

$$\int \overline{M_{ij}^{\alpha}(g)} M_{kl}^{\beta}(g) \, d\mu(g) = \frac{1}{d_\alpha} \delta_{\alpha\beta} \delta_{ik} \delta_{jl}$$

Here $d\mu(g)$ is a normalized Haar measure on G.

Proposition. *$\{\sqrt{d_\alpha} M_{ij}^\alpha(g)\}_{[\alpha]\in G;1\leq i,j\leq d_\alpha}$ form an orthonormal basis for $L^2(G\,d\mu)$.*

The Peter-Weyl theorem shows that any continuous (and as $C(G)$ is dense in $L^2(G\,d\mu)$, any square-integrable) function on G can be decomposed into joint eigenfunctions of $\mathcal{Z}(G)$.

Example. (Fourier analysis on a circle.) Let $G = \mathbf{S}^1$ be the unit circle. It is a standard result that abelian Lie groups have only one-dimensional complex representations. In case of a circle irreducible representations have form $\theta \rightarrow \exp(n\theta i)$, where θ is the polar coordinate on a circle ($\theta \in \mathbf{R}/2\pi\mathbf{Z}$).

Since all $d_\alpha = 1$, there is only one M_{ij}^α for each α. Moreover, there is a one-to-one correspondence between $\hat{G}$ and $\mathbf{Z}$, given by $[\alpha] \leftrightarrow n$ where α is given by $\theta \rightarrow \exp(n\theta i)$. With this identification $M_{ij}^n(\theta) = \exp(n\theta i)$.

The Peter-Weyl theorem says now that every continuous complex-valued function on a circle (or any continuous periodic complex-valued function on $\mathbf{R}$) may be expanded into the series in $\exp(n\theta i)$, where n varies over all $\mathbf{Z}$. Of course, this is a usual Fourier series expansion. Hence, the Peter-Weyl theorem is a generalization of familiar facts of Fourier analysis to a much wider class of compact Lie groups.

2 Laplace operator on a compact semisimple Lie group.

Among the elements of $\mathcal{Z}(G)$ there is one of particular importance, namely one that corresponds to the Casimir element of $U(\mathbf{g})$. In case of a compact semisimple Lie group it coincides with the Laplace-Beltrami operator Δ when the Riemannian structure is defined by means of the Killing form. In case of a 2-point homogenous symmetric space ($\mathbf{S}^n, \mathbf{H}^n, \mathbf{R}^n$) Δ generates $\mathcal{Z}(G)$.

2.1 Basic facts about Lie algebras and Lie groups.

Let G be a Lie group. This means that G is a smooth manifold, with the group multiplication and the inverse map which are also smooth. The tangent space at the identity e of G can be identified with a Lie algebra $\mathbf{g}$, called the Lie algebra of G.

Definition. The *adjoint* map $Ad : G \to Aut(\mathbf{g})$ is defined by

$$Ad(g)X = \frac{d}{dt}(ge^{tX}g^{-1})\bigg|_{t=0},$$

for $g \in G$ and $X \in \mathbf{g}$

The derivative of the adgoint map $Ad : G \to Aut(\mathbf{g})$ is denoted $ad : \mathbf{g} \to End(\mathbf{g})$ and is given by

$$ad(X)Y = [X, Y],$$

where $[\cdot, \cdot]$ is the Lie bracket in $\mathbf{g}$.

A maximal abelian subalgebra $\mathbf{h}$ of $\mathbf{g}$ is called a *Cartan subalgebra.* Cartan subalgebras are the Lie algebras of abelian subgroups of G called *maximal tori.*

Suppose we have chosen a maximal torus H with the corresponding Cartan subalgebra $\mathbf{h}$. Then

$$\{ad(iA)|A \in \mathbf{g}\}$$

forms an algebra of commuting self-adjoint (with respect to Killing form) linear operators on the complexified Lie algebra $\mathbf{g_C}$.

Their one-dimensional simultaneous eigenspaces are spanned by the *root elements* X_α and the eigenvalues are given by the linear functionals $\alpha(X)$ called the *roots*, so that

$$ad(iA) = \alpha(A)X_\alpha \text{ for all } A \in \mathbf{g}$$

There is a specific subset of the set of all roots, the *fundamental system.* It spans $\mathbf{h}^*$ – the dual of the vector space $\mathbf{h}$.

It can be shown that H is a flat torus, so $H = \mathbf{h}/K$ for some K isomorphic to $\mathbf{Z}^r$ ($r = dim(H)$ – the *rank* of G). Let

$$\mathcal{K} = \{X \in \mathbf{h} | e^X = id\}$$

be the "lattice of fundamental domains of H". Then the dual lattice

$$\mathcal{Y} = \{\lambda \in \mathbf{h}^* | \lambda(X) \in 2\pi\mathbf{Z} \forall X \in \mathcal{K}\}$$

is called the lattice of *weights.*

The "wedge" of weights

$$\{\lambda \in \mathcal{Y} | \langle\lambda, \alpha_j\rangle \geq 0 \text{ for all fundamental } \alpha_j\}$$

is called *fundamental weights.* It is a fundamental fact of the representation theory of compact semisimple Lie groups that the irreducible complex representations of G are in one-to-one correspondence with the set of fundamental weights.

2.2 Killing form.

One of the first facts one learns about complex representations of compact Lie groups is that it is possible to restrict attention only to unitary representations. For every complex vector space V on which a representation is considered ($A : G \to Aut(V)$) one can find an inner product, such that every $A(g)$ is unitary. The adjoint action of a compact Lie group on its Lie algebra induces an action on the complexification of Lie algebra, whence there exists an inner product on a complexified Lie algebra for which every $Ad(g)$ is unitary. It turns out that Ad is a real representation of G, and therefore there exists an inner product on the Lie algebra itself (not complexified) in which adjoint representation is unitary.

In case of a compact semisimple Lie group there is an explicit construction for such an inner product using the Killing form. It is worth pointing out that the resulting inner product is unique with respect to the properties in the proposition below only when the Lie algebra of a group is simple, i.e. does not have any proper ideals.

Definition. Let $\mathbf{g}$ be an arbitrary Lie algebra. The *Killing form* on $\mathbf{g} \times \mathbf{g}$ is defined by

$$K(X, Y) = tr(ad(X) \circ ad(Y)),$$

where $\circ$ means a composition of linear operators on $\mathbf{g}$.

Proposition. *(i) K is a symmetric bilinear form on $\mathbf{g}$;*
(ii) $ad(X)$ is skew symmetric with respect to K, i.e.

$$K(ad(X)Y, Z) = -K(Y, ad(X)Z)$$

for all $X, Y, Z \in \mathbf{g}$;
(iii) $Ad(g)$ is an isometry with respect to K, i.e.

$$K(Ad(g)X, Ad(g)Y) = K(X, Y)$$

for all $X, Y \in \mathbf{g}$ and $g \in G$.

Theorem. *Let $\mathbf{g}$ be the Lie algebra of a compact semisimple Lie group G. Then the killing form $K(X, Y)$ is strictly negative definite on $\mathbf{g}$, that is for any $X \in \mathbf{g}$, $K(X, X) < 0$ if $X \neq 0$.*

So, form the proposition and the theorem above it is clear that $\langle X, Y\rangle = -K(X, Y)$ for $X, Y \in \mathbf{g}$ defines an inner product on $\mathbf{g}$ which makes an adjoint action unitary.

Now using left translations this inner product can be extended to the whole tangent bundle of G to introduce a Riemannian structure.

Once a Riemannian structure is defined, one can construct the Laplace-Beltrami operator.

2.3 Laplace-Beltrami operator on a Riemannian manifold.

The standard Laplacian on $\mathbf{R}^n$ is defined as

$$\Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \cdots - \frac{\partial^2}{\partial x_n^2}$$

with the geometer's sign choice. Now the goal is to define an analogous object on an arbitrary Riemannian manifold. It turns out that the resulting differential operator encodes much of geometric and topological information about the underlying manifold. In case of a compact semisimple Lie group, The Laplace-Beltrami operator is bi-invariant, hence the harmonic analysis tools discussed in the Section 0.1 are applicable.

The first question that arises here is how to find an equivalent coordinate-free definition. A simple computation shows that $\Delta = -\mathrm{div} \circ \mathrm{grad}$ on $\mathbf{R}^n$. Now the question is how to generalize the operations of grad and div.

Let M be a Riemannian manifold with the Riemannian metric g. Then for $f \in C^\infty(M)$, the operation $f \to df$ produces a one-form from a function. In presence of metric there is a one-to-one correspondence between $\Omega^1(M)$ and $\Gamma(TM)$, i.e. between the space of smooth one forms and smooth vector fields. Now *define* the gradient ∇f of a smooth function f to be the element of $\Gamma(TM)$, that corresponds to df:

$$g(X, \nabla f) = df(X) \quad \text{for all } X \in \Gamma(TM)$$

From the above description it easily follows that in coordinates

$$\nabla f = g^{ij}(\partial_i f)\partial_j,$$

where $g^{ij} = (g_{ij})^{-1}$, $g_{ij} = g(\partial_i, \partial_j)$.

Generalization of div is a bit more involved. First, recall that for $f \in C_c^\infty(\mathbf{r}^n)$ and X a smooth vector field on $\mathbf{R}^n$,

$$\int_{\mathbf{R}^n} f \partial_i X^i = -\int_{\mathbf{R}^n} (\partial_i f) X^i,$$

where we apply Einstein's rule of summation over repeating indexes.

So the divergence $\partial_i X^i$ of a vector field X^i on $\mathbf{R}^n$ is characterized by the equation

$$(-\text{div}X, f) = (\nabla f, X).$$

Here the inner product $(\cdot,\cdot)$ has different meaning on the right hand side and on the left hand side. On the left it is a usual L^2 inner product, whereas on the right it is defined by $(X, Y) = \int_{\mathbf{R}^n} X \cdot Y$ for all smooth vector fields X and Y, and $X \cdot Y$ is a standard dot product on $\mathbf{R}^n$.

The last equality suggests that one can see $-\text{div}$ as a formal adjoint to ∇. Generalizing this idea to an arbitrary Riemannian manifold, by a tedious but straightforward computation one gets:

$$\text{div}X = \frac{1}{\sqrt{\det g}} \partial_i (X^i \sqrt{\det g})$$

Now the coordinate free definition of *Laplacian* Δ is

$$\Delta = -\text{div} \circ \nabla$$

There is an alternative way to define the Laplace-Beltrami operator that can be generalized an operator on all of $\Omega(M)$, that is on all smooth differential forms on M, where M is an arbitrary compact orientable Riemannian manifold.

Suppose M is an orientable n-dimensional manifold and suppose the orientation is chosen and fixed. Then, given the Riemannian structure, one has an inner product on a tangent space TM_p at every point $p \in M$. This inner product induces an inner product on T^*M_p in an obvious fashion. Now let $dx_1, \ldots, dx_n$ be an ordered basis of one-forms at some coordinate neighborhood of some point $p \in M$, having the right orientation. Then it can be easily checked that the linear extension of

$$(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}, dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_k}) =$$
$$\det \begin{pmatrix} (dx_{i_1}, dx_{j_1}) & \ldots & (dx_{i_1}, dx_{j_n}) \\ \ldots & \ldots & \ldots \\ (dx_{i_n}, dx_{j_1}) & \ldots & (dx_{i_n}, dx_{j_n}) \end{pmatrix}$$

defines an inner product on $\wedge^k T^*M_p$.

Definition. The *Hodge star operator* is a linear operator that for every k-form $\omega \in \Omega^k(M)$ produces an $n-k$ form $*\omega \in \Omega^{n-k}(M)$, according to the following formula:

$$\mu(p) \wedge \omega(p) = (\mu(p), *\omega(p)) dx_1 \wedge \ldots \wedge dx_n, \text{ for all } \mu \in \Omega^{n-p}(M) \quad \text{and all } p \in M.$$

Now one can define an inner product on $\Omega^k(M)$ using just the Hodge star.

Proposition. *The formula*

$$(\omega, \nu) = \int_M \omega \wedge *\nu$$

defines an inner product on $\Omega^k(M)$.

(Note that $\omega(p) \wedge *\nu(p) = (\omega(p), \nu(p)) dx_1 \wedge \ldots \wedge dx_n$ is an n-form, so the integration over M in the right hand side is valid).

Now we have a linear operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$, and we have defined a scalar product on both the domain and the range of it. Therefore

δ, a formal adjoint to d can be defined now:

$$\begin{aligned}(df,\omega) &= \int_M df \wedge *\omega = \int_M d(f*\omega) - (-1)^0 \int_M fd(*\omega) \\ &= -\int_M fd(*\omega) = -\int_M f**d(*\omega) = (f, -*d*\omega)\end{aligned}$$

where $\omega \in \Omega^1(M)$.

Note that since $d*\omega$ is an n-form, $**d*\omega = d*\omega$. Both f and $*d*\omega$ in the right hand side are smooth functions, that is elements of $\Omega^0(M)$, so the inner product defined above for $\Omega^k(M)$ works in this case as well, being the usual L^2 inner product.

Now, in this new notation $\delta = -*d*$ and the Laplace-Beltrami operator on functions is given by $\Delta = (-1)^n * d * d$.

2.4 The Casimir element.

From now on, assume G is a compact semisimple Lie group. It is time to benefit from the special structure of these Riemannian manifolds. Recall that $\mathcal{Z}(G)$ is the algebra of all bi-invariant differential operators on G. Let $D \in \mathcal{Z}(G)$. Now suppose we have an irreducible representation π of G on some vector space V, i.e. $\pi : G \to Aut(V)$ and $\pi(gh) = \pi(g) \circ \pi(h)$. Any representation π of a group induces a representation $\bar{\pi}$ of it's Lie algebra $\bar{\pi} : \mathbf{g} \to End(V)$. It can be proven that there exists an extension $\tilde{\pi}$ of $\bar{\pi}$ to the universal enveloping algebra $U(\mathbf{g})$ $\tilde{\pi} : U(\mathbf{g}) \to End(V)$.(See, for instance [11]) So, since $D \in U(\mathbf{g})$, there exists $\tilde{\pi}(D) \in End(V)$.

Now $\tilde{\pi}(Ad(g)D) = Ad(\pi(g))\tilde{\pi}(D)$ for all $g \in G$, but since D is bi-invariant, $D = Ad(g)D$. So, $\tilde{\pi}(D) = Ad(\pi(g))\tilde{\pi}(D)$, or

$$\tilde{\pi}(D) = \pi(g)\tilde{\pi}(D)(\pi(g))^{-1},$$

for all $g \in G$. By assumption π is irreducible, $\tilde{\pi}(D)$ commutes with $\pi(g)$ for all $g \in G$, hence by the Schur's lemma $\tilde{\pi}(D) = p_D E_V$, where E_V is an identity element of $End(V)$.

In case of the compact semisimple Lie group the irreducible representations can be labeled by their highest weights λ. Clearly p_D depends on the irreducible representation, moreover it turns out to be a polynomial in λ denoted $p_D(\lambda)$.

Consider the following sequence of standard homomorphisms:

$$Hom(\mathbf{g}, \mathbf{g}) \to \mathbf{g} \otimes \mathbf{g}^* \to \mathbf{g} \otimes \mathbf{g} \to U(\mathbf{g}).$$

Here $\mathbf{g}^*$ is a dual space to $\mathbf{g}$, and the first map comes from the canonical isomorphism. The isomorphism between $\mathbf{g}$ and $\mathbf{g}^*$ requires an inner product, in this case – the Killing form.

Definition. The Casimir element C of $U(\mathbf{g})$ is an image of $id : \mathbf{g} \to \mathbf{g}$ under this sequence of homomorphisms.

If $X_1, \ldots, X_n$ is some basis of $\mathbf{g}$, then

$$C = \sum_{i,j} g^{ij} \tilde{X}_i \tilde{X}_j,$$

and $\tilde{X}$ denotes a first order left-invariant operator corresponding to the left-invariant vector field X. Clearly if the basis is orthonormal, $C = (\tilde{X}_1)^2 + \ldots + (\tilde{X}_n)^2$.

2.5 The Casimir element of SU(2).

SU(2) is the group of all unitary 2×2 matrices A that in addition have the property $\det A = 1$. So

$$\mathrm{SU}(2) = \{A \,|\, AA^* = A^*A = I, \quad \det A = 1\},$$

where A is a 2x2 matrix and $A^* = (\bar{A})^T$.

The Lie algebra of SU(2) is denoted $\mathbf{su}(2)$ and consists of 2x2 matrices X with $trX = 0$ and $X^* = -X$. It is easy to check that the generic 2x2 matrix with these properties is given by

$$X = \begin{pmatrix} xi & -y+zi \\ y+zi & -xi \end{pmatrix} = x \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + y \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + z \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

In other words,

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

form a basis of $\mathbf{su}(2)$. A maximal torus of SU(2) from now on is chosen to be

$$H = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mid \theta \in \mathbf{R} \right\}.$$

Its Lie algebra, called a *Cartan subalgebra* is

$$\mathbf{h} = \left\{ \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix} \mid \theta \in \mathbf{R} \right\}.$$

Note that X_1 spans a Cartan subalgebra $\mathbf{h}$.

Every $ad(X_i)$ is a linear operator on $\mathbf{su}(2)$, so it may be written as a matrix in X_1, X_2, X_3 basis:

$$ad(X_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}, \qquad ad(X_2) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix},$$

$$ad(X_3) = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Now, since we know that $-K(X, Y)$ defines an inner product on $\mathbf{su}(2)$, we can find the matrix g_{ij}

$$g_{ij} = (X_i, X_j) = -K(X_i, X_j) = -tr(ad(X_i) \circ ad(X_j)).$$

For the chosen basis X_1, X_2, X_3

$$g_{ij} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Obviously,

$$g^{ij} = \frac{1}{8}\delta_{ij},$$

and therefore the Casimir element of $U(\mathbf{su}(2))$ is

$$C = 1/8(X_1^*)^2 + 1/8(X_2^*)^2 + 1/8(X_3^*)^2$$

Since SU(2) is a compact semisimple Lie group the exponential map is onto. This means that one can use the coordinates of elements of $\mathbf{su}(2)$ in the basis X_1, X_2, X_3 as global coordinates(except at the cut locus, which in this case contains only one point $g = -I$). Every function $f \in C^\infty(\mathrm{SU}(2))$ induces a function $F = F(x_1, x_2, x_3)$. Then the Casimir operator in this coordinate system looks

$$C = \left(\frac{1}{8}\frac{\partial^2}{\partial x_1^2} + \frac{1}{8}\frac{\partial^2}{\partial x_2^2} + \frac{1}{8}\frac{\partial^2}{\partial x_3^2}\right)$$

2.6 Eigenvalues of Laplacian on a compact semisimple Lie group.

In this section we will introduce a simple formula that allows one to compute the eigenvalue of the Laplacian corresponding to a given highest weight λ (and hence to an eigenfunction $\chi_\lambda(x)$).

The following theorem gives a complete description of the polynomial $p_\Delta(\lambda)$ introduced in the section 1.1.3 for the special case of D being the Laplace operator Δ.

Theorem. *The polynomial $p_\Delta(\lambda)$ is*

$$p_\Delta(\lambda) = \|\lambda + \delta\|^2 - \|\delta\|^2,$$

where λ is a maximal weight, δ is one-half the sum of positive roots, and $\|\|$ is a Killing form norm.

For proof see [2].

Example. On SU(2) with the standard choice of the maximal torus the maximal weights are numbered by positive integers and the eigenvalues of the Laplacian are

$$\lambda_n = \frac{1}{8}((n+1)^2 - 1) = \frac{n(n+2)}{8}, \quad n \in \mathbf{Z}^+$$

2.7 Heat kernel on a Riemannian manifold.

Let M be a compact, connected, oriented Riemannian manifold.

Theorem. ([9]) *There exists a heat kernel* $\rho(t,x,y) \in C^\infty(\mathbf{R}^+ \times M \times M)$ *such that*

$$(\partial_t - \Delta_x)\rho(t,x,y) = 0 \text{ for all } y \in M$$
$$\lim_{t\to 0} \textstyle\int_M \rho(t,x,y) f(y)\, dy = f(x)$$

for all $f \in C^\infty(M)$. Δ_x *denotes Laplacian acting in the* x *variable.*

So the heat kernel is a fundamental solution of the heat equation $\partial_t u = \Delta u$ on M.

When the manifold M is compact, there exists a nice expansion of the heat kernel in terms of eigenfunctions of the Laplace operator.

Theorem. (Hodge theorem for functions.) *Let* M *be a compact connected oriented Riemannian manifold. There exists an orthonormal basis of* $L^2(M)$ *consisting of eigenfunctions of the Laplacian. All the eigenvalues are positive, except that zero is an eigenvalue with multiplicity one. Each eigenvalue has finite multiplicity, and eigenvalues accumulate only at infinity.*

Suppose $\{\varphi_i(x)\}$ form an orthonormal basis of $L^2(M)$ with $\Delta\varphi_i = -\lambda_i\varphi_i$.

Proposition.

$$\rho(t,x,y) = \sum_i e^{-\lambda_i t}\varphi_i(x)\varphi_i(y).$$

Proof. Fix t and x. Now $\rho(t,x,y) = \sum_i f_i(t,x)\varphi_i(y)$ with equality in $L^2(M)$ sense. Thus

$$f_i(t,x) = \int_M \rho(t,x,y)\varphi_i(y)\,dy,$$

where dy denotes the Riemannian volume element dvol_y. From this formula it also follows that $f_i(t,x)$ is smooth in both t and x. Now

$$\begin{aligned}\partial_t f_i(t,x) &= \textstyle\int_M [\Delta_y \rho(t,x,y)]\varphi_i(y)\,dy = \int_M \rho(t,x,y)\Delta_y\varphi_i(y)\,dy \\ &= -\lambda_i \textstyle\int_M \rho(t,x,y)\varphi_i(y)\,dy = -\lambda_i f_i(t,x),\end{aligned}$$

here the symmetry $\rho(t,x,y) = \rho(t,y,x)$ coming from the self-adjointness of Laplacian is used. So,

$$\partial_t f_i(t,x) = e^{-\lambda_i t} k_i(x),$$

which means that

$$f_i(t,x) = e^{-\lambda_i t} k_i(x).$$

Therefore $\rho(t,x,y) = \sum_i e^{-\lambda_i t} k_i(x)\varphi_i(y)$, so for any $f(x) \in L^2(M)$

$$\begin{aligned}f(x) &= \lim_{t\to 0} \textstyle\int_M \rho(t,x,y) f(y)\,dy = \lim_{t\to 0} \int_M \sum_i e^{-\lambda_i t} k_i(x)\varphi_i(y) f(y)\,dy \\ &= \lim_{t\to 0} \textstyle\sum_i e^{-\lambda_i t} k_i(x) a_i = \sum_i a_i k_i(x),\end{aligned}$$

where $a_i = \int_M f(y)\varphi_i(y)\,dy$. Applied to $f(x) = \varphi_i(x)$, the last formula implies $k_i(x) = \varphi_i(x)$, and therefore $\rho(t,x,y) = \sum_i e^{-\lambda_i t}\varphi_i(x)\varphi_i(y)$ in $L^2(M)$. It follows that there exist a sequence i_j such that

$$\sum_{j=0}^{k} e^{-\lambda_{i_j} t}\varphi_{i_j}(x)\varphi_{i_j}(y) \to \rho(t,x,y)$$

pointwise for all t, x and almost all y.

By Parseval's equality,

$$(\rho(t/2,x,z), \rho(t/2,y,z)_z = \sum_i e^{-\lambda_i t/2}\varphi_i(x) e^{-\lambda_i t/2}\varphi_i(y) = \sum_i e^{-\lambda_i t}\varphi_i(x)\varphi_i(y)$$

and so $\sum_i e^{-\lambda_i t}\varphi_i(x)\varphi_i(x')$ converges pointwise everywhere with the limit continuous in t, x, y by the previous thorem. Hence $\sum_i e^{-\lambda_i t}\varphi_i(x)\varphi_i(y) \to \rho(t,x,y)$ pointwise everywhere.$\square$

2.8 Heat kernel on a compact semisimple Lie group.

In case of the underlying manifold being a compact semisimlpe Lie group, it is enough to study the heat kernel $\rho_t(x)$ based at the identity e of the group:

$$\rho_t(x) = \rho(t, e, x).$$

So, by the representation derived in the previous section,

$$\rho_t(x) = \sum_i e^{-\lambda_i t}\phi_i(e)\phi_i(x)$$

We have seen in the section 1.3 that the set of matrix elements of irreducible representations $\sqrt{d_\alpha}M_{ij}^\alpha(x)$ forms an orthonormal basis of $L^2(G)$, consisting of eigenfunctions of Laplacian (since they are eigenfunctions of any bi-invariant differential operator on G). A different version of the Weyl theorem from the section 1.3 says that the *characters* of irreducible representations $\chi_\alpha(x)$(traces of matrices $Mij^\alpha(x)$) span the subspace of $L^2(G)$ consisting of *class functions.* A function $f(g)$ on G is called a class function if it is invariant under conjugation (i.e. $f(hgh^{-1}) = f(g)$ for all $h, g \in G$. A simple computation shows that $\phi_i(x)$ in the above expression for $\rho_t(x)$ are in fact the characters $\chi_\lambda(x)$. In the semisimple Lie group case, the irreducible representations can be labeled by the highest weight λ, which is an element of certain lattice in the vector space dual to the Cartan subalgebra **h** of the Lie algebra **g**of G.

Obviously, $\chi_\lambda(e) = tr(E_{d_\lambda}) = d_\lambda$, where d_λ is the dimension of the vector space on which the corresponding representation acts.

Combining this with the results of section 2.5, we finally get a formula for the heat kernel

$$\rho_t(x) = \sum_{\lambda \in Y_d} e^{-(||\delta+\lambda||^2 - ||\delta||^2)t} d_\lambda \chi_\lambda(x)$$

Now one can using Weyl formulas for d_λ and $\chi_\lambda(x)$ (see, for instance [7])

$$d_\lambda = \frac{\pi(\lambda + \delta)}{\pi(\delta)};$$

and

$$\chi_\lambda(X) = \frac{A_{\delta+\lambda}(X)}{A_\delta(X)},$$

where

$$\pi(\mu) = \prod_{\alpha\in P+} (\alpha, \mu),$$

$$A_\nu(X) = \sum_{S\in W} (-1)^S e^{iS(\nu)(X)}, \quad \nu \in H^*,$$

X is a "logarithm" of $x \in G$ in a chosen maximal torus H, δ is one half of the sum of positive roots, and W is the Weyl group of G.

A straightforward computation shows that

$$A_\delta(X) = \prod_{\alpha\in P+} \sin(\frac{\alpha(X)}{2})$$

Finally the heat kernel has the following expression:

$$\rho_t(x) = \sum_{\lambda\in Y_d} e^{-(\|\delta+\lambda\|^2 - \|\delta\|^2)t} \frac{\prod_{\alpha\in P^+}(\alpha, \lambda+\delta)}{\prod_{\alpha\in P^+}(\alpha, \delta)} \frac{A_{\delta+\lambda}(X)}{\prod_{\alpha\in P^+}\sin(\frac{\alpha(X)}{2})}.$$

Example. (The heat kernel on SU(2).)

$$\rho_t(h) = \sum_{k=0}^{\infty} (k+1) \exp\left(-\frac{(k+2)kt}{8}\right) \frac{\sin(k+1)\theta}{\sin\theta},$$

where

$$h = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

is an element of the standard maximal torus.

Now we have arrived to the point where we can ask the main question: given this formula for the heat kernel, what can one say about the character of its critical points? The partial answer comes from the fact that $\rho_t(x)$ is a smooth class function. Chapter 3 is dedicated to the study of critical loci of smooth class functions on compact semisimple Lie groups.

Another possible direction of study comes from availability of the Poisson summation formula on a compact semisimple Lie group (see [10]and [1]). Appendix A describing the Poisson summation on flat tori prepares the ground for a more involved compact semisimple Lie group case.

3 Critical points of smooth class functions.

The goal of this chapter is to show that smooth class functions automatically have zero derivatives in certain directions on the set of singular elements of a compact connected semisimple Lie group. More specifically, if the point x is singular, i.e. lies in more then one maximal torus, the derivative of any smooth class function at x in the direction orthogonal to the line of intersection of tori is zero. This, in particular, immediately implies that all points of the center $Z(G)$ of G (which in case of a compact semisimple Lie group is finite) are critical for all smooth class functions on G.

From now on we assume that the function f is smooth and G is a compact connected semisimple Lie group. We also are going to abuse notation systematically by using the same letter for an element of the Lie algebra of a group and for the corresponding left-invariant vector field. We denote the adjoint representation of G as $Ad : G \to Aut(\mathbf{g})$ and the corresponding adjoint representation of its Lie algebra $\mathbf{g}$ as $ad : \mathbf{g} \to End(\mathbf{g})$.

Definition. A function $f : G \to \mathbf{R}(\mathbf{C})$ is called a *class function* if $f(hgh^{-1}) = f(g)$ for all $g, h \in G$.

The following proposition provides the main tool for this chapter. Finding the right elements to conjugate with is all what is needed to establish the announced properties of derivatives of class functions.

Proposition. *For a smooth class function f,*

$$Xf(x) = (Ad(g)X)f(gxg^{-1}) \qquad (1),$$

where $x, g \in G$ and $X \in \mathbf{g}$. $Ad(g)$ denotes the adjoint action of G on its Lie algebra $\mathbf{g}$.

Proof.

$$\begin{aligned} Xf(x) &= \tfrac{d}{dt} f(xe^{tX})|_{t=0} = \tfrac{d}{dt} f(gxe^{tX}g^{-1})|_{t=0} \\ &= \tfrac{d}{dt} f(gxg^{-1}ge^{tX}g^{-1})|_{t=0} = (Ad(g)X)f(gxg^{-1}) \square \end{aligned}$$

In particular this formula implies that the set of critical points of f is closed under conjugation as follows. If x is a critical point of f then for all X in $\mathbf{g}$,

$$(Ad(g)X)f(gxg^{-1}) = (Xf)(x) = 0,$$

and so gxg^{-1} is also a critical point since $Ad(g)$ is an invertible linear operator.

Observe that if there is an element $g \in G$ so that g commutes with x (i.e. $gxg^{-1} = x$) and $Ad(g)X = -X$ for some $X \in \mathbf{g}$, it follows from the proposition that

$$Xf(x) = (Ad(g)X)f(gxg^{-1}) = -Xf(g),$$

which means that $Xf(g) = 0$. It is this observation that allows us to establish the desired properties of smooth class functions, provided we can find the element g with the specified properties.

Let H be a maximal torus of G and let P be the set of roots and P^+ be the set of positive roots of G. Note $P \subset \mathbf{h}^*$, where $\mathbf{h}^*$ is the dual of the Lie algebra of H.

For every $\alpha \in P^+$ one can define three vectors $\tau_\alpha \in i\mathbf{h}$, $X_\alpha, X_{-\alpha} \in \mathbf{g_C}$, where $\mathbf{g_C}$ is the complexification of $\mathbf{g}$. Those are the appropriately normalized (with respect to the Killing form inner product) *root vector* τ_α (the dual of the linear functional $\alpha(\bullet) \in \mathbf{h}^*$) and two *root elements* for α and $-\alpha$ respectively. Each of the root elements X_α and $X_{-\alpha}$ spans a *root space* — the simultaneous eigenspace of the family $\{ad(X) | X \in i\mathbf{h}\}$ corresponding to the given root. Here $ad(X)$ denotes the adjoint action of the Lie algebra of G on itself).

The following relations hold for these three vectors:

$$[X_\alpha, X_{-\alpha}] = \tau_\alpha,$$

$$[\tau_\alpha, X_{\pm\alpha}] = \pm 2X_{\pm\alpha},$$

$$X_{-\alpha} = \overline{X_\alpha}.$$

Vectors $X_\alpha + X_{-\alpha}$ and $i(X_\alpha - X_{-\alpha})$ lie in $\mathbf{g} \subset \mathbf{g_C}$ and span the two-dimensional real eigenspace $\mathbf{m}_\alpha$ that corresponds to the pair of roots α and $-\alpha$; $i\tau_\alpha$ lies in $\mathbf{h}$. Thus one can decompose $\mathbf{g}$ as follows:

$$\mathbf{g} = \mathbf{h} \oplus \bigoplus_{\alpha \in P^+} \mathbf{m}_\alpha$$

Root vectors lie in $\mathbf{h}$ and $rank(G)$ of them form a *fundamental system* are also a basis for $\mathbf{h}$.

Hence, $(i/2)\tau_1, \ldots, (i/2)\tau_r$ (normalized root vectors of the fundamental system, $r = rank(G)$) and $(1/2)(X_\alpha + X_{-\alpha}), (i/2)(X_{-\alpha} - X_\alpha)$ for all $\alpha \in P^+$ form a basis of $\mathbf{g}$.

Denote

$$C_\alpha = \frac{i}{2}\tau_\alpha,$$

$$A_\alpha = \frac{i}{2}(X_{-\alpha} - X_\alpha), \qquad B_\alpha = \frac{1}{2}(X_\alpha + X_{-\alpha}).$$

It is easy to check that the triple $(A_\alpha, B_\alpha, C_\alpha)$ satisfies

$$[A_\alpha, B_\alpha] = C_\alpha, \qquad [B_\alpha, C_\alpha] = A_\alpha, \qquad [C_\alpha, A_\alpha] = B_\alpha. \tag{2}$$

Now $C_1, \ldots, C_r$ and the set of A_α, B_α for all positive roots α form a basis of $\mathbf{g}$.

Lemma. *Let G be a Lie group and $\mathbf{g}$ be its Lie algebra. If the triple (A, B, C) of elements $A, B, C \in \mathbf{g}$, satisfies*

$$[A, B] = C \qquad [B, C] = A \qquad [C, A] = B, \tag{2'}$$

then

$$Ad(e^{\theta A})C = \cos(\theta)C - \sin(\theta)B \tag{3}$$

for all $\theta \in \mathbf{R}$.

Proof. By a standard fact of the theory of Lie groups that

$$Ad(e^X) = e^{ad(X)},$$

(see, for an instance, [11] Theorem 2.13.2), where

$$e^{ad(X)} = I + ad(X) + \frac{1}{2}[ad(X)]^2 + \ldots + \frac{1}{n!}[ad(X)]^n + \ldots.$$

Therefore

$$\begin{aligned} Ad(e^{\theta A})C &= C + \theta[A, C] + \tfrac{\theta^2}{2}[A, [A, C]] + \tfrac{\theta^3}{3!}[A, [A, [A, C]]] + \ldots \\ &= C - B - \tfrac{\theta^2}{2}C + \tfrac{\theta^3}{3!}[A, -C] + \ldots \\ &= C - B - \tfrac{\theta^2}{2}C + \tfrac{\theta^3}{3!}B + \ldots \end{aligned}$$

It is not very hard to see that $\{[ad(A)]^n C\}$ produces a cyclic sequence $-B, -C, B, C, -B, -C, B, C \ldots$, and hence the right hand side of the last formula may be written as

$$\sum_{n=0}^{\infty} \frac{\theta^{2n}}{(2n)!}(-1)^n C - \sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!}(-1)^n B = \cos(\theta)C - \sin(\theta)B. \square$$

It turns out that the formula (3) gives one a great amount of information about derivatives of smooth class functions on G.

Theorem. *Let f be a smooth class function on a compact connected semisimple Lie group G and let H be a maximal torus in G. Then for all $x \in H$ and all $\alpha \in P^+$,*

$$A_\alpha f(x) = B_\alpha f(x) = 0.$$

Proof. It is easy to check that both $(C_\alpha, A_\alpha, B_\alpha)$ and $(C_\alpha, -B_\alpha, A_\alpha)$ satisfy relations (2'), so it follows from (3) that

$$Ad(e^{\theta C_\alpha})B_\alpha = \cos(\theta)B_\alpha - \sin(\theta)A_\alpha \tag{4'}$$

and

$$Ad(e^{\theta C_\alpha})A_\alpha = \cos(\theta)A_\alpha + \sin(\theta)B_\alpha. \tag{4''}$$

Choosing $\theta = \pi$, one gets

$$Ad(e^{\pi C_\alpha})B_\alpha = -B_\alpha, \qquad Ad(e^{\pi C_\alpha})A_\alpha = -A_\alpha. \tag{5}$$

But, C_α lies in $\mathbf{h}$, therefore $\exp(\pi C_\alpha) \in H$, which means that it commutes with any $x \in H$. Combining (1) and (5) one gets

$$A_\alpha f(x) = (Ad(e^{\pi C_\alpha})A_\alpha) f(e^{\pi C_\alpha} x e^{-\pi C_\alpha}) = -A_\alpha f(x),$$

so $A_\alpha f(x) = 0$ and similarly $B_\alpha f(x) = 0$ for all $x \in H$. □

Definition. For $\alpha \in P$ define

$$U_\alpha = \{x \in H \mid Ad(x)Y = Y \text{ for all } Y \in \mathbf{g}_\alpha\},$$

where $\mathbf{g}_\alpha$ is the root space corresponding to α.

Theorem. ([7], Theorem VIII.7.2) *U_α is a closed subgroup of G and obeys*

$$U_\alpha = \{e^X \mid X \in \mathbf{h}, \alpha(X) \in 2\pi i\mathbf{Z}\}.$$

The Lie algebra of U_α is

$$\pi_\alpha = \{X \in H \mid \alpha(X) = 0\}.$$

One of the fundamental properties of a compact semisimple Lie groups is that every element of such group belongs to a maximal torus (see [7]).

Definition. An element of G is *singular* if it lies in more than one maximal torus.

Theorem. ([7], Section VIII.7)

(i) $x \in H$ *is regular if and only if it lies in no* U_α*'s;*

(ii) *if* $x \in H$ *lies in exactly* k *distinct* U_α *then*

$$dim N(\{x\}) = l + 2k,$$

where $l = rank(G)$ *and* $N(\{x\}) = \{g \in G \,|\, gxg^{-1} = x\}$ *is the normalizer of* x *in* G*;*

(iii) $Z(G) = \bigcap_{\alpha \in P} U_\alpha$*, where* $Z(G) = \{g \in G \,|\, gh = hg$ *for all* $h \in G\}$ *is the center of* G*.*

Theorem. *Let* G *be a compact connected semisimple Lie group,* H *be a maximal torus,* P *be the roots of* G *and* f *be a smooth class function on* G*. Then for all* $x \in U_\alpha$

$$C_\alpha f(x) = 0.$$

Proof. Let $x \in U_\alpha$ for some $\alpha \in P$. We know from (3) that

$$Ad(e^{\pi A_\alpha})C_\alpha = -C_\alpha,$$

thus if we can prove that x commutes with $\exp(\pi A_\alpha)$, it would follow from (1) that

$$C_\alpha f(x) = -C_\alpha f(x)$$

and therefore $C_\alpha f(x) = 0$.

Recall: $N(\{x\}) = \{y \in G \,|\, xyx^{-1} = y\}$, so the Lie algebra of $N(\{x\})$ is $\mathbf{n}_x = \{X \in \mathbf{g} \,|\, Ad(x)X = X\}$.

Since $x \in U_\alpha$, $Ad(x)X_\alpha = X_\alpha$ and

$$Ad(x)X_{-\alpha} = Ad(x)\overline{X_\alpha} = \overline{Ad(x)X_\alpha} = X_{-\alpha}.$$

Here $\overline{X}$ denotes complex conjugation and the fact that $Ad(x)$ is a real linear operator on $\mathbf{g}$ is used.

So, $Ad(x)A_\alpha = A_\alpha$, and therefore $\pi A_\alpha \in \mathbf{n}_x$, which means $\exp(\pi A_\alpha) \in N(\{x\})$. $\square$

Corollary. *If $x \in Z(G)$, x is a critical point of every smooth class function on G.*

Proof. Let $x \in Z(G)$ and let f be a smooth class function. For all $\alpha \in P$ $x \in U_\alpha$, $C_\alpha f(x)$ is zero for all $\alpha \in P$. Since for every smooth class function $A_\alpha f(x) = B_\alpha f(x) = 0$, the derivative of f in all basic directions are zero at x. Therefore x is a critical point.□

Corollary. *If $x \in U_\alpha$ for all U_α corresponding to some fundamental system then x is a critical point of any smooth class function.*

Proof. C_α corresponding to a fundamental system form a basis for **h**, therefore again derivatives in all basic directions are zero. □

Note that the Lie algebra of U_α is orthogonal to τ_α (and therefore to C_α) at every point $x \in U_\alpha$. That is for all $X \in \pi_\alpha$, $\alpha(X) = 0$, which means $(\tau_\alpha, X) = 0$, by definition of τ_α.

Thus, the last theorem says that that the derivative of a class function is necessarily zero in the direction orthogonal to U_α at every $x \in U_\alpha$.

Summarizing, the derivative of an arbitrary smooth class function is only nonzero in the directions belonging to the Cartan subalgebra (Lie algebra of a maximal torus) for a regular element x of G. If the point x is singular but not central, the derivative is zero in some of those directions, namely in ones that are orthogonal to U_α to which x belongs, and may be nonzero in the others. Finally, for a central point x, all derivatives are zero so it is always a critical point.

Now suppose x is a critical point of a smooth class function f, and $x \notin Z(G)$. As we have noticed above, the critical set of f is closed under conjugation, which implies that all points in $\{yxy^{-1} \,|\, y \in G\}$ are critical.

Proposition. *If x is a critical point of a smooth class function f and $x \notin Z(G)$ then the points of G conjugate to x form a submanifold of critical* *points of f. The dimension of this submanifold is* $\dim G - \dim N(\{x\})$.

Proof. First we observe that G acts on itself by conjugation, and that $N(\{x\})$ is a stabilizer of x with respect to this action. Now application of Theorem 2.9.7 and Corollary 2.9.8 from [11] gives us the desired result. □

We have already seen that $\dim N(\{x\}) = l + 2k$, where l is the rank of G and k is the number of different U_α containing x. This means that the more singular is x, that is the more U_α contain it, the smaller is the corresponding critical submanifold.

Another simple implication of this result is that if f has at least one noncentral critical point, then it can't be a Morse function, since it has a whole submanifold of critical points.

From the Remark 1 after the Theorem 4.1 in [6], we know that the only case when a smooth function on some manifold can have exactly two critical points (no matter degenerate or not) is when the manifold is homeomorphic to a sphere. But we also know that $Sp(n)$ and $Spin(2n+1)$ have centers isomorphic to $\mathbf{Z}_2$ (see [7] p.151) As $Sp(n)$ and $Spin(2n+1)$ are not homeomorphic to spheres (see, for an instance, [3]), no class function on these groups are Morse.

Moreover, most classical groups have too small a center in order for the smooth class function to have only central critical points and satisfy Morse inequalities. So, in general, a smooth class function is not going to be Morse.

Since the heat kernel $\rho_t(x)$ (by *the* heat kernel we mean the one based at the identity element of the group) on a compact semisimple Lie group G is a smooth class function, all previous analysis is applicable.

In particular, we know that all points in the center of G are critical points of $\rho_t(x)$, and that one should not expect $\rho_t(x)$ to be a Morse function on G. For an instance, $\rho_t(x)$ is certainly not a Morse function on G_2, on $Sp(n)$, and on $Spin(2n+1)$.

Now, when we know that the heat kernel is not Morse, the next natural question to ask is if it is Morse-Bott, that is if the critical locus is a submanifold and the Hessian of the heat kernel on it is not degenerate in the directions transversal to the cut locus. The answer is unknown, but "yes" seems quite expectable.

Another question of interest is if the critical locus of the heat kernel (at least at the small time limit) is related to the cut locus of G. The famous Varadan's result on the short time limit of the heat kernel suggests that such a relation is not impossible.

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Appendix A. Poisson summation on a flat torus.

The availability of Poisson summation on the standard torus $\mathbf{T}^n = \mathbf{R}^n/2\pi\mathbf{Z}^n$ is well-known. Details can be found in Chapter VII of [8]. It is clear that generalization of this process to an arbitrary flat torus should not be very hard. This appendix contains a detailed construction of Poisson summation of a flat skew torus.

Definition. Let V be a real vector space, $n = \dim V$, let $X_1, \dots, X_n$ be a basis of V and let

$$\Gamma = \{k_1 X_1 + \ldots + k_n X_n \,|\, (k_1, \dots, k_n) \in \mathbf{Z}^n\}$$

be a lattice in V corresponding to this basis. Then we call $\mathbf{T}^n_\Gamma = V/\Gamma$ an *n-dimensional skew torus.*

It is clear from the definition that geometrically and topologically one may think of $\mathbf{T}^n_\Gamma$ as of the "parallelogram"

$$D = \{\lambda_1 X_1 + \ldots + \lambda_n X_n \,|\, 0 \leq \lambda_i \leq 1, 1 \leq i \leq n\},$$

with opposite sides identified. D is a *fundamental domain* of Γ.

First we need the analogue of Fourier series. Recall that

$$\Gamma^* = \{m \in V^* \,|\, m(\gamma) \in \mathbf{Z} \quad \text{for all } \gamma \in \Gamma\}$$

is called the *dual lattice* for Γ.

Proposition. *The set of functions* $\exp(il(x))$*, for all* $l \in 2\pi\Gamma^*$ *forms an orthonormal basis of* $L^2(\mathbf{T}^n_\Gamma, d\mu)$*, where* $d\mu = (1/\mathrm{vol}(D))dx_1 \dots dx_n$ *is the normalized Haar measure.*

Proof. $\mathbf{T}^n_\Gamma$ is an abelian group with the operation being vector space addition $\mathrm{mod}\Gamma$ in V, so all of it irreducible unitary complex representations are one-dimensional. Hence the characters of irreducible representations have the form $\chi_\alpha(X) = \exp(il(X))$ for each class of irreducible representations α, where $l(X)$ is some real-valued function.

Now, $\chi_\alpha(X+Y) = \chi_\alpha(X)\chi_\alpha(Y)$, since the representations are one-dimensional, and so $l(X)$ has to be linear.

By the definition of the group operation, $l(X+\gamma) = l(x)$ for all $\gamma \in \Gamma$, thus $l \in 2\pi\Gamma^*$.

So, we have shown that all characters of irreducible representations of $\mathbf{T}^n_\Gamma$ have the required form. It is easy to see that conversely, all functions of the form $\exp(il(X))$ with $l \in 2\pi\Gamma^*$ are representations of $\mathbf{T}^n_\Gamma$. Unitarity and irreducibility follow immediately. □

Definition. Let $F : V \to \mathbf{C}$, then its *Fourier transform* is defined as

$$\hat{F}(l) = \frac{1}{(2\pi)^n} \int_V F(x) e^{il(x)}\, dx.$$

Theorem.(Poisson summation on a torus.) *Let V be a finite dimensional vector space ($n = \dim V$), Γ be a lattice in V and $\mathbf{T}^n_\Gamma = V/\Gamma$. If $F : V \to \mathbf{C}$ is continuous and its Fourier transform $\hat{F}(l) : V^* \to \mathbf{C}$ is also continuous, and if*

$$|F(x)| \le \frac{A}{(1+|x|)^{n+\delta}}; \quad \textit{and} \quad |\hat{F}(l)| \le \frac{B}{(1+|l|)^{n+\delta}},$$

for some $A, B, \delta > 0$, then

$$\sum_{\gamma\in\Gamma} F(x+\gamma) = \frac{(2\pi)^n}{\mathrm{vol}(D)} \sum_{l\in 2\pi\Gamma^*} \hat{F}(l) e^{il(x)}.$$

Proof. Let

$$f(x) = \sum_{\gamma\in\Gamma} F(x+\gamma).$$

From the bounds on $F(x)$ it follows that the series converges absolutely. Clearly $f(x)$ is a function on $\mathbf{T}^n_\Gamma = V/\Gamma$, so we can find its Fourier coefficients:

$$c_l = \frac{1}{\mathrm{vol}(D)} \int_D f(x) e^{-il(x)}\, dx,$$

for $l \in 2\pi\Gamma^*$.

Using the definition of $f(x)$ and the bounds on $F(x)$,

$$c_l = \frac{1}{\mathrm{vol}(D)} \sum_{\gamma\in\Gamma} \int_D F(x+\gamma)e^{-il(x)}\,dx = \frac{1}{\mathrm{vol}(D)} \sum_{\gamma\in\Gamma} \int_{D+\gamma} F(y)e^{-il(y)}\,dy,$$

the fact that $l \in 2\pi\Gamma^*$ makes the last step possible. Now the last expression is a sum of integrals over disjoint copies of D that tile V, so we get

$$c_l = \frac{1}{\mathrm{vol}(D)} \int_V F(y)e^{-il(y)}\,dy = \frac{(2\pi)^n}{\mathrm{vol}(D)} \hat{F}(l)$$

Now, from the bound on $\hat{F}(l)$ we see that $\sum |c_l|$ converges and therefore by the Corollary 1.8 in Chapter 5 of [8] the sum

$$\sum_{l\in 2\pi\Gamma^*} c_l e^{il(x)} = \sum_{l\in 2\pi\Gamma^*} \frac{(2\pi)^n}{\mathrm{vol}(D)} \hat{F}(l) e^{il(x)}$$

converges to $f(x)$ on $\mathbf{T}^n_\Gamma = V/\Gamma$, and hence the Poisson summation formula holds. □